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Reflection of high-frequency elastic waves from a non-plane boundary surface of the elastic medium

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Abstract

The paper is concerned with a classical problem of reflection of a high-frequency (longitudinal or transverse) wave, generated by a point source located in the elastic medium, by a free non-plane smooth boundary surface of this medium. For the investigation of this problem, we develop a method founded on the estimate of reflection integrals by the two-dimensional stationary phase method. The proposed approach permits derivation of the amplitude of reflected longitudinal and transverse waves in explicit form. The amplitudes of the reflected waves are defined by principal curvatures, by Gaussian curvature of the boundary surface at the point of specular reflection, by the distance from the source and receiver to the point of specular reflection, by direction of the incident wave, and by elastic moduli.

1. Introduction

In the present paper we study the problem of reflection by an arbitrarily shaped smooth boundary surface of the elastic body, in the case when a (longitudinal or transverse) elastic wave falls to the boundary from a point source placed in the medium. The process is assumed to be harmonic in time: $\mathbf{u}(x, y, z, t) = \mathrm{Re}[\mathbf{u}(x, y, z)\exp(-\mathrm{i}\omega t)]$, and the boundary of the elastic body is stress-free. Here $\mathbf{u}$ is the displacement vector.

Practical applications of this theoretical problem are urgent, first of all, in ultrasonic testing of materials. Assume a void defect located in an elastic sample. A standard method of detection is an insonification of the defect by high-frequency ultrasonic waves of various incidence. To adequately estimate how the reflected amplitude is connected with geometry of the flaw, one should develop analytical formulas for these amplitudes. This is well described in the case of plane reflecting surfaces [1,2], and the main goal of the present paper is to extend the classical results to the case of non-plane surfaces.

In the two-dimensional case a solution to this problem was obtained in Ref. [1]. In the three-dimensional scalar acoustic problem an explicit-form high-frequency representation for the acoustic pressure, in the case of a single reflection from an arbitrary smooth surface, can be found in Refs. [2,3].

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In the present work, we propose a method to study reflection of longitudinal and transverse waves by the boundary surface of a linear homogeneous isotropic elastic body, which is founded upon the estimate of reflection integrals by the two-dimensional stationary phase method. All over below we imply that the boundary conditions on the reflecting surfaces correspond to zero normal and tangential stresses.

2. Spherical incident wave of longitudinal type: *p–p* reflection

First of all, let us give a general representation of the solution in terms of boundary integrals, which will be used below for all types of wave reflections and wave transformations.

Let a point source x_0 in the elastic medium generate a spherical high-frequency wave, which is incident to a boundary surface. It is known [2] that the amplitude of the reflected high-frequency wave at the point x is defined by the direction of incidence and by a small vicinity S of the point y^* of specular (mirror) reflection on the boundary surface. Therefore, with increasing frequency the amplitude of the reflected wave can be obtained in frames of geometrical ray theory.

If the ray x_0–y–x is reflected from the surface S, $(y \in S)$ only once (see Fig. 1), then the displacement vector in the wave, reflected from the free surface, is determined by the following integral [4]:

$$u_k(x) = \iint_S \mathbf{T}_y[\mathbf{U}^{(k)}(y,x)] \cdot \mathbf{u}(y)\,\mathrm{d}S_y, \quad U_j^{(k)}(y,x) = U_{j_p}^{(k)}(y,x) + U_{j_s}^{(k)}(y,x),$$

$$U_{j_p}^{(k)}(y,x) = -\frac{1}{4\pi\rho\omega^2}\frac{\partial^2}{\partial y_k \partial y_j}\left(\frac{\mathrm{e}^{\mathrm{i}k_p R}}{R}\right) \quad (k,j = 1,2,3),$$

$$U_{j_s}^{(k)}(y,x) = \frac{1}{4\pi\rho\omega^2}\left[\delta_{kj}k_s^2\frac{\mathrm{e}^{\mathrm{i}k_s R}}{R} + \frac{\partial^2}{\partial y_k \partial y_j}\left(\frac{\mathrm{e}^{\mathrm{i}k_s R}}{R}\right)\right], \quad R = |\mathbf{y} - \mathbf{x}|,$$

$$\mathbf{T}_y[\mathbf{U}^{(k)}(y,x)] = 2\mu\frac{\partial \mathbf{U}^{(k)}}{\partial n} + \lambda\,\mathbf{n}\cdot \mathrm{div}(\mathbf{U}^{(k)}) + \mu\mathbf{n}\times \mathrm{curl}(\mathbf{U}^{(k)}). \quad (1)$$

Here $\mathbf{U}^{(k)}(y,x)$, $\mathbf{T}_y[\mathbf{U}^{(k)}(y,x)]$, $k = 1,2,3$ are Green's tensors of displacement and stress, respectively, $\mathbf{u}(y)$ is the vector of full displacement field on the boundary surface, ρ is the mass density, λ, μ are the Lamè coefficients, $k_p = \omega/c_p, k_s = \omega/c_s$ are the longitudinal and transverse wavenumbers, and c_p, c_s are the respective wave speeds, $\mathbf{n}$ is the outward normal to the surface S at the point y, δ_{kj} is the Kronecker delta.

Let us assume that the incident wave is determined by a harmonically oscillating point force applied at the point x_0. This force may generate a spherical wave, which in the direction of propagation $\mathbf{q} = (\mathbf{y} - \mathbf{x_0})/|\mathbf{y} - \mathbf{x_0}|$

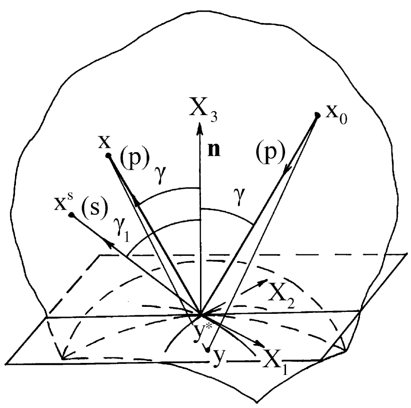

Fig. 1. Reflection of the longitudinal wave from a non-plane surface.

can be both longitudinal and transverse:

$$\mathbf{u}_p^{\mathrm{inc}} = Q\frac{\mathrm{e}^{\mathrm{i}k_pR_0}}{R_0}\mathbf{q}, \quad \mathbf{u}_s^{\mathrm{inc}} = Q\frac{\mathrm{e}^{\mathrm{i}k_sR_0}}{R_0}\mathbf{q}_1, \quad R_0 = |\mathbf{y}-\mathbf{x_0}|, \tag{2}$$

where vector $\mathbf{q}_1$ is perpendicular to $\mathbf{q}$. Here Q is the modulus of projection of the applied point force in direction $\mathbf{q} = \{-\cos\alpha, -\cos\beta, -\cos\gamma\}$, in the case of longitudinal incident wave, and $\mathbf{q}_1 = \{-\cot\gamma\cos\alpha, -\cot\gamma\cos\beta, \sin\gamma\}$, in the case of transverse incident wave.

Let us consider p–p reflection of the incident longitudinal p-wave, given by the first expression in Eq. (2), its p–s transformation to the transverse s-wave will be considered in Section 3. Let us relate a small vicinity of the point $y^* \in S$ to the right Cartesian coordinate system $OX_1X_2X_3$, which are determined by the unit normals and by the tangents to the curvature lines of the surface at the point $y^* \in S$. In the chosen coordinate system the unit normal $\mathbf{n}$ to the surface at the point y^* has the coordinates $\mathbf{n} = \{0, 0, 1\}$.

Let us study here in more detail the p–p reflection. In this case the coordinates of the displacement vector in the reflected p-wave are

$$u_k^{(p)}(x) = \iint_S \left[\mu\sum_{m=1}^{2}\left(\frac{\partial U_{m_p}^{(k)}}{\partial y_3} + \frac{\partial U_{3_p}^{(k)}}{\partial y_m}\right)u_m(y) + \left(2\mu\frac{\partial U_{3_p}^{(k)}}{\partial y_3} + \lambda\,\mathrm{div}\mathbf{U}_p^{(k)}\right)u_3(y)\right]\mathrm{d}S_y. \tag{3}$$

The representations derived below give the leading asymptotic terms of respective formulas at high frequencies.

In order to asymptotically estimate integral (3) as $k_p \to \infty$, let us use the following asymptotic representations:

$$\mathrm{div}\,\mathbf{U}_p^{(k)}(y,x) = \frac{\mathrm{i}k_p^3}{4\pi\rho\omega^2}\frac{\mathrm{e}^{\mathrm{i}k_pR}}{R}\frac{\partial R}{\partial y_k}\left[1 + O\left(\frac{1}{k_p}\right)\right],$$

$$\frac{\partial U_{j_p}^{(k)}}{\partial y_m} = \frac{\mathrm{i}k_p^3}{4\pi\rho\omega^2}\frac{\mathrm{e}^{\mathrm{i}k_pR}}{R}\frac{\partial R}{\partial y_m}\frac{\partial R}{\partial y_k}\frac{\partial R}{\partial y_j}\left[1 + O\left(\frac{1}{k_p}\right)\right], \quad k,j,m = 1,2,3,$$

$$x = (x_1,x_2,x_3), \quad y = (y_1,y_2,y_3),\ (y \in S), \quad \frac{\partial R}{\partial y_1} = \frac{y_1 - x_1}{r} = -\cos\alpha,$$

$$\frac{\partial R}{\partial y_2} = \frac{y_2 - x_2}{r} = -\cos\beta, \quad \frac{\partial R}{\partial y_3} = \frac{y_3 - x_3}{r} = \cos\gamma \quad (k_p \to \infty). \tag{4}$$

Here $\{-\cos\alpha, -\cos\beta, \cos\gamma\}$ are direction cosines of vector $\mathbf{y} - \mathbf{x}$.

Now, by substituting Eq. (4) to Eq. (3), we obtain

$$u_k^{(p)}(x) = \frac{\mathrm{i}k_p^3}{4\pi\rho\omega^2}\iint_S \Phi(y)\frac{\partial R}{\partial y_k}\frac{\mathrm{e}^{\mathrm{i}k_pR}}{R}\,\mathrm{d}S_y,$$

$$\Phi(y) = \left[\frac{\partial R}{\partial y_1}u_1(y) + \frac{\partial R}{\partial y_2}u_2(y)\right]\frac{\partial R}{\partial y_3} + \left[2\mu\left(\frac{\partial R}{\partial y_3}\right)^2 + \lambda\right]u_3(y). \tag{5}$$

Let us pass to a spherical coordinate system (r, θ, ψ) with the center at the point y^*. The components of the displacement vector can be written as follows:

$$u_r^{(p)}(x) = \frac{\mathrm{i}k_p^3}{4\pi\rho\omega^2}\iint_S \Phi(y)\frac{\mathrm{e}^{\mathrm{i}k_pR}}{R}\,\mathrm{d}S_y, \quad u_\theta^{(p)}(x) = 0, \quad u_\psi^{(p)}(x) = 0,$$

$$\Phi(y) = -2\mu[\cos\alpha u_1(y) + \cos\beta u_2(y)]\cos\gamma + (2\mu\cos^2\gamma + \lambda)u_3(y). \tag{6}$$

When estimating asymptotics of Kirchhoff's integral in formula (6), the components of the full displacement field $u_k(y)$, $k = 1,2,3$ under the integral should be taken as a solution to a local problem on reflection of a plane incident p-wave from a plane boundary of the elastic half-space (see, for instance, Ref. [5]):

$$u_m(y) = \left[1 + V_{pp}(y) - \frac{k_s}{k_p \sin\gamma}\sqrt{1 - \frac{k_p^2}{k_s^2}\sin^2\gamma}\, V_{ps}(y)\right] u_{m_p}^{\mathrm{inc}}(y), \quad m = 1, 2,$$

$$u_3(y) = [1 - V_{pp}(y) - \tan\gamma\, V_{ps}(y)] u_{3_p}^{\mathrm{inc}}(y), \tag{7}$$

where V_{pp} and V_{ps} are the coefficients of p–p reflection and p–s transformation [5]:

$$V_{pp} = \frac{4\cot\gamma\cot\gamma_1 - (1 - \cot^2\gamma_1)^2}{z}, \quad V_{ps} = \frac{4\cot\gamma(1 - \cot^2\gamma_1)}{z},$$

$$z = 4\cot\gamma\cot\gamma_1 + (1 - \cot^2\gamma_1)^2. \tag{8}$$

By substituting Eqs. (7) and (2) into Eq. (6), we come to the following integral representation of the radial displacement:

$$u_r^{(p)}(x) = \frac{Q\mathrm{i}k_p^3}{4\pi k_s^2}\iint_S \left\{ -\sin 2\gamma\left[-\sin\gamma(1 + V_{pp}) + \frac{k_s}{k_p}\sqrt{1 - \frac{k_p^2}{k_s^2}\sin^2\gamma}\, V_{ps}\right] \right.$$
$$\left. + \left(\frac{k_s^2}{k_p^2} - 2\sin^2\gamma\right)[-\cos\gamma(1 - V_{pp}) + \sin\gamma\, V_{ps}]\right\} \frac{\mathrm{e}^{\mathrm{i}k_p(R_0+R)}}{R_0 R}\,\mathrm{d}S_y. \tag{9}$$

If we substitute relations (8) into the integrand of the last expression, we can analytically prove that the complex structure arising there can be simplified as

$$\frac{k_p^2}{2k_s^2}\left\{ -\sin 2\gamma\left[-\sin\gamma(1 + V_{pp}) + \frac{k_s}{k_p}\sqrt{1 - \frac{k_p^2}{k_s^2}\sin^2\gamma}\, V_{ps}\right] \right.$$
$$\left. + \left(\frac{k_s^2}{k_p^2} - 2\sin^2\gamma\right)[-\cos\gamma(1 - V_{pp}) + \sin\gamma\, V_{ps}]\right\} = \cos\gamma\, V_{pp}. \tag{10}$$

The obtained equality allows us to derive the following basic representation for $u_r^{(p)}(x)$, after taking non-oscillating functions of the integrand (in the high-frequency approximation) out of the sign of the integral:

$$u_r^{(p)}(x) = \frac{Q\mathrm{i}k_p}{2\pi}\frac{\cos\gamma}{L_0 L} V_{pp}(y^*)\iint_S \mathrm{e}^{\mathrm{i}k_p\varphi}\,\mathrm{d}S_y,$$

$$\varphi = |\mathbf{y} - \mathbf{x_0}| + |\mathbf{y} - \mathbf{x}|, \quad L_0 = |\mathbf{y}^* - \mathbf{x_0}|, \quad L = |\mathbf{y}^* - \mathbf{x}|. \tag{11}$$

Ray representation can be obtained from Eq. (11), by using a multidimensional stationary phase method [6]. In the introduced coordinate system with the center at the point y^*, arbitrary point $y \in S$ from a vicinity of y^* has locally the coordinates $y = \{\Delta s_1, \Delta s_2, -0.5[k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2]\}$, where Δs_1, Δs_2 are small differentials along the curvature lines, $k_1 = 1/R_1$ and $k_2 = 1/R_2$ are principal curvatures, R_1 and R_2 are principal curvature radii of the surface S at the point $y^* \in S$, $[k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2]$ is the second quadratic form of the surface at the point y^*, if one relates the surface to the curvature lines.

Let us apply the cosine theorem to triangles x_0–y^*–y and x–y^*–y:

$$|x_0 - y|^2 = L_0^2 + |\mathbf{\Delta s}|^2 - 2L_0|\mathbf{\Delta s}|\cos(x_0 y^* \widehat{\ } y^* y),$$

$$|x - y|^2 = L^2 + |\mathbf{\Delta s}|^2 - 2L|\mathbf{\Delta s}|\cos(x y^* \widehat{\ } y^* y). \tag{12}$$

By calculating the scalar product of the vector $\{\cos\alpha, \cos\beta, \cos\gamma\}$ (the unit vector in direction $\mathbf{x_0} - \mathbf{y}^*$) with the vector $\mathbf{\Delta s} = \{\Delta s_1, \Delta s_2, -0.5[k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2]\}$, and the scalar product of the vector

$\{-\cos\alpha, -\cos\beta, \cos\gamma\}$ (the unit vector in direction $\mathbf{x}-\mathbf{y}^*$) with the vector $\mathbf{\Delta s}$, we can deduce

$$|\mathbf{\Delta s}|\cos(x_0y^*\widehat{\ }y^*y) = 0.5[k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2]\cos\gamma + \Delta s_1\cos\alpha + \Delta s_2\cos\beta,$$

$$|\mathbf{\Delta s}|\cos(xy^*\widehat{\ }y^*y) = 0.5[k_1(\Delta s_1)^2 + k_2(\Delta s_2)^2]\cos\gamma - \Delta s_1\cos\alpha - \Delta s_2\cos\beta. \tag{13}$$

If we neglect the terms of smaller orders compared with $(\Delta s_1)^2$, $\Delta s_1\Delta s_2$, $(\Delta s_2)^2$, then formula (12) leads to the representations

$$\begin{aligned}|x_0 - y| &= L_0 - \Delta s_1\cos\alpha - \Delta s_2\cos\beta + 0.5(L_0^{-1}\sin^2\alpha + k_1\cos\gamma)(\Delta s_1)^2\\ &\quad - L_0^{-1}\cos\alpha\cos\beta\Delta s_1\Delta s_2 + 0.5(L_0^{-1}\sin^2\beta + k_2\cos\gamma)(\Delta s_2)^2,\end{aligned}$$

$$\begin{aligned}|x - y| &= L + \Delta s_1\cos\alpha + \Delta s_2\cos\beta + 0.5(L^{-1}\sin^2\alpha + k_1\cos\gamma)(\Delta s_1)^2\\ &\quad - L^{-1}\cos\alpha\cos\beta\Delta s_1\Delta s_2 + 0.5(L^{-1}\sin^2\beta + k_2\cos\gamma)(\Delta s_2)^2.\end{aligned} \tag{14}$$

Consequently,

$$\varphi = L_0 + L + 0.5d_{11}(\Delta s_1)^2 + d_{12}\Delta s_1\Delta s_2 + 0.5d_{22}(\Delta s_2)^2,$$

$$d_{11} = (L_0^{-1} + L^{-1})\sin^2\alpha + 2k_1\cos\gamma, \quad d_{12} = -(L_0^{-1} + L^{-1})\cos\alpha\cos\beta,$$

$$d_{22} = (L_0^{-1} + L^{-1})\sin^2\beta + 2k_2\cos\gamma. \tag{15}$$

Note that the first powers of Δs_1 and Δs_2 are not present in the phase φ. This confirms that the point y^* of the direct ray reflection corresponds to a stationary value of the phase function φ. The leading asymptotic term of integral (11) is thus determined by the coefficients in front of $(\Delta s_1)^2, \Delta s_1\Delta s_2, (\Delta s_2)^2$, and can be derived from Eq. (11) by using the two-dimensional stationary phase method [6], as follows:

$$u_r^{(p)}(x) = QV_{pp}(y^*)\cos\gamma\,\frac{\exp\{\mathrm{i}[k_p(L_0 + L) + \pi(\delta_2^{(pp)} + 2)/4]\}}{L_0L\sqrt{|\det[\mathbf{D_2^{(pp)}}]|}}, \tag{16}$$

where $\mathbf{D_2^{(pp)}}$ is the Hessian of the symmetric structure: $d_{ij} = d_{ji},\ i,j = 1,2$, and $\delta_2^{(pp)} = \mathrm{sgn}[\mathbf{D_2^{(pp)}}]$ is the difference between the number of positive and negative eigenvalues of this symmetric matrix $\mathbf{D_2^{(pp)}}$.

The final result is

$$u_r^{(p)}(x) = \frac{QV_{pp}(y^*)\exp\{\mathrm{i}[k_p(L_0 + L) + \pi(\delta_2^{(pp)} + 2)/4]\}}{\sqrt{\left|(L_0 + L)^2 + 2L_0L(L_0 + L)\dfrac{k_2\sin^2\alpha + k_1\sin^2\beta}{\cos\gamma} + 4L_0^2L^2K\right|}}. \tag{17}$$

Here $K = k_1k_2$ is the Gaussian curvature of the surface S at the point y^*, and vector $\{-\cos\alpha, -\cos\beta, -\cos\gamma\}$ determines direction of the incident ray $x_0 - y^*$ in the chosen coordinate system.

In this formula one can outline the two simple particular cases. First of all, if $k_1 = k_2 = 0$ then the well-known result for the reflection from a free plane boundary follows from (17)

$$u_r^{(p)}(x) = -QV_{pp}\frac{\exp[\mathrm{i}k_p(L_0 + L)]}{L_0 + L}. \tag{18}$$

Another interesting case is related to a back reflection (when $V_{pp} = -1$) in a far zone. In this case expression (17) reduces to the form

$$u_r^{(p)}(x) = -0.5Q\mathrm{i}\frac{\sqrt{R_1R_2}}{L_0^2}\exp[\mathrm{i}(2k_pL_0 + \pi\delta_2^{(pp)}/4)]. \tag{19}$$

It should also be noted that asymptotic estimate holds in the case when a (high-frequency) longitudinal wave falls to a convex side of the boundary of elastic medium. If the wave is incident to a concave surface then the principal curvatures k_1 and k_2 have to be taken negative.

It is also noteworthy that formula (17) differs from analogous result for acoustic pressure $p(x)$ in the reflected wave of the scalar theory only by the reflection coefficient V_{pp}, which is present in the elastic case.

3. Spherical incident wave of longitudinal type: *p–s* transformation

Let us study the transformation of the incident *p*-wave to the reflected *s*-one. In this case the Cartesian coordinates of the displacement vector in the reflected *s*-wave $u_k^{(s)}(x)$, $k=1,2,3$ at the point x are

$$u_k^{(s)}(x) = \mu \iint_S \left[\sum_{m=1}^{2} \left(\frac{\partial U_{m_s}^{(k)}}{\partial y_3} + \frac{\partial U_{3_s}^{(k)}}{\partial y_m} \right) u_m(y) + 2 \frac{\partial U_{3_s}^{(k)}}{\partial y_3} u_3(y) \right] \mathrm{d}S_y. \tag{20}$$

In order to estimate this integral, let us apply the asymptotic representation of the following functions:

$$\frac{\partial U_{j_s}^{(k)}}{\partial y_m} = \frac{\mathrm{i}k_s^3}{4\pi\rho\omega^2} \left(\delta_{kj} - \frac{\partial R}{\partial y_k} \frac{\partial R}{\partial y_j} \right) \frac{\partial R}{\partial y_m} \frac{\mathrm{e}^{\mathrm{i}k_s R}}{R} \left[1 + O\left(\frac{1}{k_s}\right) \right], \quad k,j,m = 1,2,3,$$

$$x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3), \ (y \in S), \quad \frac{\partial R}{\partial y_1} = \frac{y_1 - x_1}{r} = -\cos\alpha_1,$$

$$\frac{\partial R}{\partial y_2} = \frac{y_2 - x_2}{r} = -\cos\beta_1, \quad \frac{\partial R}{\partial y_3} = \frac{y_3 - x_3}{r} = \cos\gamma_1 \quad (k_s \to \infty). \tag{21}$$

Here $\{-\cos\alpha_1, -\cos\beta_1, \cos\gamma_1\}$ are direction cosines of vector $\mathbf{x}-\mathbf{y}$.

By substituting Eq. (21) to Eq. (20), we deduce

$$u_1^{(s)}(x) = \frac{\mathrm{i}\mu k_s^3}{4\pi\rho\omega^2} \iint_S \left\{ \left[1 - 2\left(\frac{\partial R}{\partial y_1}\right)^2 \right] u_1(y) - 2\frac{\partial R}{\partial y_1}\frac{\partial R}{\partial y_2} u_2(y) - 2\frac{\partial R}{\partial y_1}\frac{\partial R}{\partial y_3} u_3(y) \right\} \times \frac{\partial R}{\partial y_3} \frac{\mathrm{e}^{\mathrm{i}k_s R}}{R} \mathrm{d}S_y. \tag{22a}$$

$$u_2^{(s)}(x) = \frac{\mathrm{i}\mu k_s^3}{4\pi\rho\omega^2} \iint_S \left\{ -2\frac{\partial R}{\partial y_1}\frac{\partial R}{\partial y_2} u_1(y) + \left[1 - 2\left(\frac{\partial R}{\partial y_2}\right)^2 \right] u_2(y) - 2\frac{\partial R}{\partial y_2}\frac{\partial R}{\partial y_3} u_3(y) \right\} \times \frac{\partial R}{\partial y_3} \frac{\mathrm{e}^{\mathrm{i}k_s R}}{R} \mathrm{d}S_y. \tag{22b}$$

$$u_3^{(s)}(x) = \frac{\mathrm{i}\mu k_s^3}{4\pi\rho\omega^2} \iint_S \left\{ \left[1 - 2\left(\frac{\partial R}{\partial y_3}\right)^2 \right] \left[\frac{\partial R}{\partial y_1} u_1(y) + \frac{\partial R}{\partial y_2} u_2(y) \right] + 2\left[1 - \left(\frac{\partial R}{\partial y_3}\right)^2 \right] \frac{\partial R}{\partial y_3} u_3(y) \right\} \times \frac{\mathrm{e}^{\mathrm{i}k_s R}}{R} \mathrm{d}S_y. \tag{22c}$$

Let us pass again to the spherical coordinate system (r, θ, ψ) at the point y^*. Then the components of the displacement vector can be reduced to the form

$$u_\theta^{(s)}(x) = \frac{\mathrm{i}\mu k_s^3}{4\pi\rho\omega^2} \iint_S \frac{1}{\sin\gamma_1} \{ -\cos 2\gamma_1 [\cos\alpha_1 \, u_1(y) + \cos\beta_1 \, u_2(y)] - 2\sin^2\gamma_1 \cos\gamma_1 u_3(y) \} \frac{\mathrm{e}^{\mathrm{i}k_s R}}{R} \mathrm{d}S_y, \quad u_r^{(s)}(x) = 0, \quad u_\psi^{(s)}(x) = 0. \tag{23}$$

Similarly to the case of *p–p* reflection, in the asymptotic estimate of the last integral we accept the components of the displacement vector on the surface S in the form of Eqs. (7)–(8).

After substitution of Eqs. (7)–(8) into Eq. (23) and taking into account the identity

$$- \cos 2\gamma_1 \left[- \sin\gamma(1+V_{pp}) + \frac{k_s}{k_p}\sqrt{1-\frac{k_p^2}{k_s^2}\sin^2\gamma V_{ps}(y)} \right] - \sin 2\gamma_1[-\cos\gamma(1-V_{pp}) + \sin\gamma V_{ps}] = -2\frac{k_s}{k_p}\cos\gamma_1 V_{ps}, \tag{24}$$

which can be proved directly, we reduce the (high-frequency) expression for the displacement $u_\theta^{(s)}(x)$ to the form

$$u_\theta^{(s)}(x) = -\frac{Q\mathrm{i}k_s^2}{2\pi k_p}\frac{\cos\gamma_1}{L_0L_1}V_{ps}(y^*)\iint_S e^{\mathrm{i}\varphi_{ps}}\,\mathrm{d}S_y,$$

$$\varphi_{ps} = k_p|\mathbf{y}-\mathbf{x_0}| + k_s|\mathbf{x}-\mathbf{y}|, \quad L_0 = |\mathbf{y}^*-\mathbf{x_0}|, \quad L_1 = |\mathbf{x}-\mathbf{y}^*|. \tag{25}$$

The first terms of power expansion in the quantities $|\mathbf{y}-\mathbf{x_0}|$ and $|\mathbf{x}-\mathbf{y}|$, present in the structure of the phase function, have the form

$$\begin{aligned}|x_0-y| = {} & L_0 - \Delta s_1\cos\alpha - \Delta s_2\cos\beta + 0.5(L_0^{-1}\sin^2\alpha + k_1\cos\gamma)(\Delta s_1)^2 \\ & - L_0^{-1}\cos\alpha\cos\beta\Delta s_1\Delta s_2 + 0.5(L_0^{-1}\sin^2\beta + k_2\cos\gamma)(\Delta s_2)^2,\end{aligned}$$

$$\begin{aligned}|y-x| = {} & L_1 + \Delta s_1\cos\alpha_1 + \Delta s_2\cos\beta_1 + 0.5(L_1^{-1}\sin^2\alpha_1 + k_1\cos\gamma_1)(\Delta s_1)^2 \\ & - L_1^{-1}\cos\alpha_1\cos\beta_1\Delta s_1\Delta s_2 + 0.5(L_1^{-1}\sin^2\beta_1 + k_2\cos\gamma_1)(\Delta s_2)^2.\end{aligned} \tag{26}$$

Let us prove that in the phase function $\varphi_{ps} = k_p|\mathbf{y}-\mathbf{x_0}| + k_s|\mathbf{x}-\mathbf{y}|$ the first powers of Δs_1 and Δs_2 are absent. These terms have the form: $(k_s\cos\alpha_1 - k_p\cos\alpha)\Delta s_1$ and $(k_s\cos\beta_1 - k_p\cos\beta)\Delta s_2$. In the studied p–s transformation the following Snell's law holds: $k_p\sin\gamma = k_s\sin\gamma_1$. Let us consider, for instance, the first term:

$$k_s\cos\alpha_1 - k_p\cos\alpha = k_p\frac{\sin\gamma}{\sin\gamma_1}\cos\alpha_1 - k_p\cos\alpha = k_p\sin\gamma\left(\frac{\cos\alpha_1}{\sin\gamma_1} - \frac{\cos\alpha}{\sin\gamma}\right). \tag{27}$$

Since the incident x_0–y^* and the reflected y^*–x rays are situated in the same plane with the normal to the surface at the point $y^* \in S$, the following relations are valid:

$$\frac{\cos\alpha}{\sin\gamma} = \frac{\cos\alpha_1}{\sin\gamma_1}, \quad \frac{\cos\beta}{\sin\gamma} = \frac{\cos\beta_1}{\sin\gamma_1}. \tag{28}$$

Consequently, coefficients in front of Δs_1 and Δs_2 are zero.

Under these conditions, the phase φ_{ps} can be reduced to the form

$$\varphi_{ps} = k_pL_0 + k_sL_1 + 0.5d_{11}(\Delta s_1)^2 + d_{12}\Delta s_1\Delta s_2 + 0.5d_{22}(\Delta s_2)^2,$$

$$d_{11} = k_pL_0^{-1}\sin^2\alpha + k_sL_1^{-1}\sin^2\alpha_1 + k_1(k_p\cos\gamma - k_s\cos\gamma_1),$$

$$d_{12} = -(k_pL_0^{-1}\cos\alpha\cos\beta + k_sL_1^{-1}\cos\alpha_1\cos\beta_1),$$

$$d_{22} = k_pL_0^{-1}\sin^2\beta + k_sL_1^{-1}\sin^2\beta_1 + k_2(k_p\cos\gamma - k_s\cos\gamma_1), \tag{29}$$

where k_1 and k_2 are the principal curvatures of the boundary surface at the reflection point.

Note once again that the first powers Δs_1 and Δs_2 are absent here that proves the point y^* of the direct ray reflection to correspond to a stationary value of the phase φ_{ps}. The leading asymptotic term of integral (25) is thus determined by the coefficients in front of $(\Delta s_1)^2$, $\Delta s_1\,\Delta s_2$, $(\Delta s_2)^2$, and can be obtained from expression (25) by the double stationary phase method [6], in the following form:

$$u_\theta^{(s)}(x) = -QV_{ps}(y^*)\cos\gamma_1 \frac{k_s^2}{k_p} \frac{\exp\{\mathrm{i}[k_pL_0 + k_sL_1 + \pi(\delta_2^{(ps)} + 2)/4]\}}{L_0L_1\sqrt{|\det[\mathbf{D}_2^{(\mathbf{ps})}]|}}, \tag{30}$$

where the elements of symmetric ($d_{12} = d_{21}$) Hessian $\mathbf{D}_2^{(\mathbf{ps})} = d_{ij},\ i,j = 1,2$ are determined by formulas (29), and $\delta_2^{(ps)} = \mathrm{sgn}[\mathbf{D}_2^{(\mathbf{ps})}]$ is the difference between the number of positive and negative eigenvalues of the matrix $\mathbf{D}_2^{(\mathbf{ps})}$.

4. Spherical incident wave of transverse type: *s–s* reflection

Let us study the wave transformations when the incident wave is transverse, being given by the second relation (2). Let the direction of the incidence be defined by the unit vector $\{-\cos\alpha_1, -\cos\beta_1, -\cos\gamma_1\}$.

The components of the displacement vector in the reflected *s*-wave at the point x can be determined again from Eqs. (22). For all that the displacement vector $\mathbf{u}(y)$ on the boundary surface is defined from the solution of a local problem on reflection of the transverse *s*-wave from the plane tangent to the surface S at the point of specular reflection y^*. Solution to this rather classical problem can be found, for example, in Ref. [5], being given as follows:

$$u_m(y) = [V_{ss}(y) - 1 - \tan\gamma_1 V_{sp}(y)]u_{ms}^{\mathrm{inc}}(y) \quad m = 1,2,$$

$$u_3(y) = \left[V_{ss}(y) + 1 + \frac{k_p}{k_s\sin\gamma_1}\sqrt{1 - \frac{k_s^2\sin^2\gamma_1}{k_p^2}}V_{sp}(y)\right]u_{3s}^{\mathrm{inc}}(y), \tag{31}$$

where V_{ss} and V_{sp} are the coefficients of *s–s* and *s–p* transformations:

$$V_{ss} = \frac{4\cot\gamma\cot\gamma_1 - (\cot^2\gamma_1 - 1)^2}{z}, \quad V_{sp} = \frac{4\cot\gamma_1(\cot^2\gamma_1 - 1)}{z},$$

$$z = 4\cot\gamma\cot\gamma_1 + (\cot^2\gamma_1 - 1)^2. \tag{32}$$

In the local spherical coordinate system (r, θ, ψ) linked to the point y^* the components of the displacement vector in the reflected *s*-wave can be expressed as

$$u_\theta^{(s)}(x) = \frac{Q\mathrm{i}k_s}{4\pi}\iint_S \Bigg\{ -\cos 2\gamma_1[\cos\gamma_1(V_{ss} - 1) - \sin\gamma_1 V_{sp}] - \sin 2\gamma_1\left[\sin\gamma_1(V_{ss} + 1) + \frac{k_p}{k_s}\sqrt{1 - \frac{k_s^2}{k_p^2}\sin^2\gamma_1 V_{sp}}\right]\Bigg\} \frac{\mathrm{e}^{\mathrm{i}k_s(R_0+R)}}{R_0R}\,\mathrm{d}S_y,$$

$$u_r^{(s)}(x) = 0, \quad u_\psi^{(s)}(x) = 0. \tag{33}$$

Having substituted relations (32) for the reflection and transmission coefficients into Eq. (33), it can be analytically proved the arising expression is

$$-\cos 2\gamma_1[\cos\gamma_1(V_{ss} - 1) - \sin\gamma_1 V_{sp}] - \sin 2\gamma_1\left[\sin\gamma_1(V_{ss} + 1) + \frac{k_p}{k_s}\sqrt{1 - \frac{k_s^2}{k_p^2}\sin^2\gamma_1 V_{sp}(y)}\right] = -2\cos\gamma_1 V_{ss}. \tag{34}$$

Taking into account this relation, integral representation (33) of the (high-frequency) asymptotic solution can be rewritten in the following form:

$$u_\theta^{(s)}(x) = -\frac{Q\mathrm{i}k_s}{2\pi}\frac{\cos\gamma_1}{L_0 L} V_{ss}(y^*) \iint_S \mathrm{e}^{\mathrm{i}k_s\varphi}\,\mathrm{d}S_y,$$

$$\varphi = |\mathbf{y}-\mathbf{x_0}| + |\mathbf{x}-\mathbf{y}|, \quad L_0 = |\mathbf{y}^*-\mathbf{x_0}|, \quad L = |\mathbf{x}-\mathbf{y}^*|. \tag{35}$$

The leading asymptotic term of $u_\theta^{(s)}(x)$ can be obtained from Eq. (35) in the same way as above for p–p and p–s transformations, by using the two-dimensional stationary phase method, that finally results in the following expression:

$$u_\theta^{(s)}(x) = -QV_{ss}(y^*)\frac{\exp\{\mathrm{i}[k_s(L_0+L)+\pi(\delta_2^{(ss)}+2)/4]\}}{L_0 L\sqrt{|\det[\mathbf{D_2^{(ss)}}]|}}, \tag{36}$$

where $\delta_2^{(ss)} = \mathrm{sgn}[\mathbf{D_2^{(ss)}}]$, and Hessian $\mathbf{D_2^{(ss)}}$ has the same structure as $\mathbf{D_2^{(pp)}}$, and this can be obtained from $\mathbf{D}_2^{\mathbf{(pp)}}$ by taking $\cos\alpha_1, \cos\beta_1, \cos\gamma_1$ instead of $\cos\alpha, \cos\beta, \cos\gamma$.

5. Incidence of the transverse spherical wave: *s–p* transformation

The components of the displacement vector in the reflected p-wave at the point x, under s–p transformation, are defined by formulas (20), where the components of vector $\mathbf{u}(y)$ on the boundary surface should be defined from solutions (31), (32) of the local problem concerning reflection of the transverse s-wave from the plane tangential to the surface S at the reflecting point y^*.

In the local spherical coordinate system (r, θ, ψ) coupled with the point y^* the only non-trivial component of the displacement vector in the reflected p-wave is the radial one:

$$\begin{aligned} u_r^{(p)}(x) = \frac{Q\mathrm{i}k_p^3}{4\pi k_s^2}\iint_S \Bigg\{ & -\sin 2\gamma[\cos\gamma_1(V_{ss}-1) - \sin\gamma_1 V_{sp}] + \left(\frac{k_s^2}{k_p^2} - 2\sin^2\gamma\right) \\ & \times \left[\sin\gamma_1(V_{ss}+1) + \frac{k_p}{k_s}\sqrt{1-\frac{k_s^2}{k_p^2}\sin^2\gamma_1}\, V_{sp}(y)\right]\Bigg\} \frac{\mathrm{e}^{\mathrm{i}(k_sR_0+k_pR)}}{R_0 R}\,\mathrm{d}S_y, \end{aligned}$$

$$u_\theta^{(p)}(x) = 0, \quad u_\psi^{(p)}(x) = 0. \tag{37}$$

Here vector $\{-\cos\alpha, -\cos\beta, -\cos\gamma\}$ determines the direction of propagation of the reflected p-wave.

By analogy to the previous cases, substitution of expressions (37) into Eq. (20) reduces the problem to a double integral, with a complex structure of the integrand. The construction arising there can be simplified as follows:

$$\begin{aligned} \frac{k_p}{2k_s}\Bigg\{ & -\sin 2\gamma[\cos\gamma_1(V_{ss}-1) - \sin\gamma_1 V_{sp}] + \left(\frac{k_s^2}{k_p^2} - 2\sin^2\gamma\right) \\ & \times \left[\sin\gamma_1(V_{ss}+1) + \frac{k_p}{k_s}\sqrt{1-\frac{k_s^2}{k_p^2}\sin^2\gamma_1}\, V_{sp}(y)\right]\Bigg\} = \cos\gamma\, V_{sp}(y) \end{aligned} \tag{38}$$

that can be proved directly.

Taking into account this relation, integral representation (37) for $u_r^{(p)}(x)$ can be simplified to the following form:

$$u_r^{(p)}(x) = \frac{Q\mathrm{i}k_p^2\cos\gamma}{2\pi k_s L_0 L} V_{sp}(y^*)\iint_S \mathrm{e}^{\mathrm{i}\varphi_{sp}}\,\mathrm{d}S_y,$$

$$\varphi_{sp} = k_s|\mathbf{y} - \mathbf{x_0}| + k_p|\mathbf{x} - \mathbf{y}|, \quad L_0 = |\mathbf{y}^* - \mathbf{x_0}|, \quad L = |\mathbf{x} - \mathbf{y}^*|,$$

$$\varphi_{sp} = k_s L_0 + k_p L + 0.5 d_{11}(\Delta s_1)^2 + d_{12}\Delta s_1 \Delta s_2 + 0.5 d_{22}(\Delta s_2)^2,$$

$$d_{11} = k_s L_0^{-1} \sin^2\alpha_1 + k_p L^{-1} \sin^2\alpha + k_1(k_s\cos\gamma_1 - k_p\cos\gamma),$$

$$d_{12} = -(k_s L_0^{-1}\cos\alpha_1\cos\beta_1 + k_p L^{-1}\cos\alpha_1\cos\beta_1),$$

$$d_{22} = k_s L_0^{-1}\sin^2\beta_1 + k_p L^{-1}\sin^2\beta + k_2(k_s\cos\gamma_1 - k_p\cos\gamma). \tag{39}$$

The leading asymptotic term, which can be obtained from Eq. (39) by the double stationary phase method, is

$$u_r^{(p)}(x) = QV_{sp}(y^*)\cos\gamma \frac{k_p^2}{k_s} \frac{\exp\{\mathrm{i}[k_s L_0 + k_p L + \pi(\delta_2^{(sp)} + 2)/4]\}}{L_0 L\sqrt{|\det[\mathbf{D_2^{(sp)}}]|}}, \tag{40}$$

where the elements of the symmetric ($d_{12} = d_{21}$) Hessian matrix $\mathbf{D_2^{(sp)}} = d_{ij},\ i,j = 1,2$ are determined by formulas (39), and $\delta_2^{(sp)} = \mathrm{sgn}[\mathbf{D_2^{(sp)}}]$.

6. Discussions and physical conclusions

The principal developed formulas (17), (30), (36), and (40) are worthy of a detailed discussion. In order to provide an alternative glance at the subject, let us rewrite these formulas in a different way. We demonstrate this idea on example of the first of them. It can be shown, based on some results of differential geometry [7,8], that Eq. (17) is equivalent to

$$u_r^{(p)}(x) = \frac{QV_{pp}(y^*)\exp\{\mathrm{i}[k_p(L_0 + L) + \pi(\delta_2^{(pp)} + 2)/4]\}}{\sqrt{\left|(L_0 + L)^2 + 2L_0 L(L_0 + L)\dfrac{2H\cos^2\gamma + \tilde{k}\sin^2\gamma}{\cos\gamma} + 4L_0^2 L^2 K\right|}}, \tag{41}$$

where $K = k_1 k_2$ is again the Gaussian curvature, $H = (k_1 + k_2)/2$ is the average curvature at the point of specular reflection y^*, and $\tilde{k}$ is the curvature of the normal section of the surface by the plane of the ray $x_0 - y^* - x$. The latter is defined by the Euler formula

$$\tilde{k} = k_1\cos^2\tilde{\varphi} + k_2\sin^2\tilde{\varphi} \quad \left(\cos\tilde{\varphi} = \frac{\cos\alpha}{\sin\gamma}, \sin\tilde{\varphi} = \frac{\cos\beta}{\sin\gamma}\right), \tag{42}$$

which represents the curvature of arbitrary normal section in terms of the principal curvatures $k_1,\ k_2$ and the angle $\tilde{\varphi}$ (the latter is the angle between the tangent to this normal section and the first principal direction).

The developed asymptotic expressions (17), (30), (36), and (40) show that the displacement amplitude of the reflected waves is defined rather complicatedly by geometric and physical parameters of the problem.

The amplitude of the reflected wave is inversely proportional to a root square in the denominator, which depends upon local geometric characteristics at the point of specular reflection y^*, upon the distance between the source and the receiver from the reflection point, by the direction of the incidence and reflection, as well as by the elastic constants.

Let us describe the features of the amplitude on examples of p–p and s–s reflections. Let us consider Eq. (41), which give the leading asymptotic term as $k_p L_0 \gg 1, k_p L \gg 1, k_p R_1 \gg 1, k_p R_2 \gg 1$ in the p–p case.

In Section 2 we outlined the two extreme cases of a locally plane reflecting surface ($k_1 = k_2 = 0$) and a far-field back reflection ($\gamma = 0, L_0 = L, L_0 \to \infty$). In the last case the amplitude of the reflected wave is determined by the distance L_0 and by the Gaussian curvature K.

Let us consider the case when the distances, L_0, L, and the principal curvatures, R_1, R_2, are of the same order. In this case all three terms in the denominator are of the same order. The second term contains the information about the shape of the surface through its average curvature, $H = (k_1 + k_2)/2$, and the curvature $\tilde{k}$ of the normal section given by the ray $x_0 - y^* - x$.

The third term provides dependence upon the Gaussian curvature K. The contribution of the second and the third terms is determined by a local shape of the surface at the point y^*. If the reflection point is elliptic and the ray is incident to a convex part of the surface $(k_1 > 0, k_2 > 0)$ all these terms are positive and the ray divergence (scattering) is maximum. If the ray is incident to a concave part $(k_1 < 0, k_2 < 0)$ then the second term is negative $(H < 0, \tilde{k} < 0, K > 0)$ and the ray divergence is less when compared with the previous case. Note that with decreasing γ the contribution of the term containing the average curvature H increases, but of the one containing $\tilde{k}$ decreases. In the case of normal incidence the second term depends on the average curvature only. With increasing γ one can observe the increase of the contribution of $\tilde{k}$.

If the reflection point y^* is hyperbolic $(K < 0)$ then the principal curvatures k_1 and k_2 have different signs. Consequently, one of the principal sections is bent to the direction of chosen normal, and the other—to the opposite direction. At such a point there exist two asymptotic directions (with $\tilde{k} = 0$) passing symmetrically with respect to the principal directions. Near the hyperbolic reflection point the surface is of saddle shape. For all that the sign of the third term with K is always negative, and the sign of the second term depends on the values of H, $\tilde{k}$, γ.

If the reflection point y^* is parabolic $(K = 0)$ then at least one value among k_1 and k_2 is zero. In the case $k_1 = k_2 = 0$ the respective result is discussed in Section 2. Let, for example, $k_1 \neq 0,\ k_2 = 0$. Then the quantities $H = k_1$ and $\tilde{k} = k_1 \cos^2 \varphi$ have the same sign as k_1, which can be either positive or negative. In the parabolic case the third term is zero, and the second one has the same sign as the non-trivial principal curvature. For all that in the case of normal incidence the geometrical properties of the surface are determined by the non-trivial curvature of the principal section, which is present in the second term as a factor.

The presence of the coefficient $V_{pp}(y^*)$ in the numerator of Eqs. (17)–(41) indicates that the qualitative properties of the reflection and mode conversion are the same as in the case of reflection from a plane tangent to the given surface at the point of specular reflection.

The main conclusions of the above consideration can be directly transferred to the case of *s–s* reflection.

In the case of *p–s* and *s–p* transformations respective formulas are more complicated being dependent upon many parameters when estimating quantitatively the reflected amplitude. The presence (as factors) of the wavenumbers k_p and k_s $(k_p < k_s)$ indicates a smaller contribution of all geometric parameters containing k_p. However, all conclusions about the influence of local geometric characteristics of the surface remain valid.

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